(7 pages) Reg. No. : ....

Code No. : 5760 **Sub. Code : WMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2024.

Second Semester

Mathematics — Core

ADVANCED ALGEBRA

(For those who joined in July 2023 onwards)

Time : Three hours Maximum : 75 marks

PART A — (15 × 1 = 15 marks)

Answer ALL questions.

Choose the correct answer :

1. Let F be a field. If a field K contains F then K is __________ of F.

   (a) subfield (b) subgroup

   (c) superset (d) extension

2. If $\dim([K:F]) = m$ then degree (K) = __________.

   (a) m (b) n

   (c) $m-1$ (d) $n-1$

3. The extension K of F is called an algebraic extension of F if every element in K __________ over F.

   (a) simple (b) finite

   (c) normal (d) algebraic

4. K is a normal extension of F if K is a finite extension of F such that F is the __________ of $G(K, F)$.

   (a) Field (b) Quotient

   (c) Fixed field (d) Subfield

5. Any finite extension of a field of characteristic __________ is a simple extension.

   (a) ∞ (b) 2

   (c) 1 (d) 0

6. The __________ of a group G is a subfield of K.

   (a) Subfield (b) Fixed field

   (c) Splitting field (d) Quotient field

7. Let F be a filed of rational numbers and $\omega = e^{\frac{2\pi i}{5}}$ then ω^5 = __________.

   (a) 0 (b) -1

   (c) 2 (d) 1

8. The automorphism σ of K is in $G(K, F)$ if $\sigma(\alpha)$ = __________.

   (a) $\alpha - 1$ (b) $\frac{1}{\alpha}$

   (c) α (d) $-\alpha$

9. The σ is an automorphism of K then the fixed field of G is the set of all $a \in K$ such that $\sigma(a)$ = __________ for all $\sigma \in G$.

   (a) a (b) $a-1$

   (c) $\frac{1}{a}$ (d) a^2

10. If the finite field F has p^m elements then every $\alpha \in F$ satisfies __________.

    (a) $a^m = a$ (b) $a^m = p$

    (c) $a^{p^m} = a$ (d) $a^p = m$

11. A complex number θ is said to be a primitive n^{th} root of unity if θ^n = __________.

    (a) 2 (b) n

    (c) 1 (d) 0

12. The polynomial $\phi_n(x) = \pi(x-\theta)$ is called a __________ polynomial.

    (a) cyclotomic (b) cubic

    (c) monic (d) quadratic

13. A group G is solvable if for $G = N_0 \supset N_1 .... \supset N_k = \{e\}$ where N_i is a normal subgroup of $N_i - 1$ and N_{i-1}/N_i is __________.

    (a) abelian (b) cyclic

    (c) prime (d) non-abelian

14. The roots of the polynomial $x^3 + 3x + 4$ over the field of rationals are __________.

    (a) $-3 \pm \sqrt{-7}$ (b) $-7 \pm \sqrt{-3}$

    (c) $\frac{-3 \pm \sqrt{-7}}{2}$ (d) $\frac{-7 \pm \sqrt{-3}}{2}$

15. Every polynomial of degree n over a field of complex numbers has all its n roots in the __________ of complex numbers.

    (a) group (b) field

    (c) fixed field (d) subfield

PART B — (5 × 4 = 20 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If a, b in k are algebraic over F then show that $a+b$, $a-b$ are algebraic over F.

Or

(b) If $a \in K$ is algebraic of degree n over F then show that $[F(a):F]=n$

17. (a) Let $f(x) \in F[x]$ be a polynomial of degree ≥ 1. then show that there is an extension E of F of degree atmost $n!$ in which $f(x)$ has n roots.

Or

(b) If F is a field of characteristic $p \neq 0$ then show that the polynomial $x^{p^m} - x \in F[x]$ has distinct roots.

18. (a) Let K be a field of complex numbers and let F be a field of real numbers. Find $G(K, F)$.

Or

(b) Let K be a Normal extension of F. Then prove that (i) $[K:K_H]=O(H)$; (ii) $H=G(K, K_H)$.

19. (a) Let F be a field with q elements and suppose that $F \subset K$ where K is also a finite field. Then show that K has q^n elements.

Or

(b) Show that the multiplicative group of non-zero elements of a finite field is cyclic.

20. (a) Prove that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

Or

(b) For all x, y in Q show that the adjoint in Q satisfies (i) $x^{**}=x$; (ii) $(xy)^* = y^* x^*$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

21. (a) If L is a finite extension of K and if K is a finite extension of F then prove that L is a finite extension of F.

Or

(b) If L is an algebraic extension of K and if K is an algebraic extension of F then prove that L is an algebraic extension of F.

22. (a) Prove that a polynomial of degree n over a field can have atmost n roots in any extension field.

Or

(b) Prove : A polynomial $f(x) \in F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a nontrivial common factor.

23. (a) If K is a finite extension of F, then prove that $G(K, F)$ is a finite group and $O(G(K, F)) \leq [K:F]$.

Or

(b) Prove that fundamental theorem of Galois theory.

24. (a) Prove that any two finite fields having the same number of elements are isomorphic.

Or

(b) State and prove the Wedderburn theorem.

25. (a) State and prove Frobenius theorem.

Or

(b) State and prove Four Square Theorem.

———————